DEFINITION 1 Let V be an arbitrary nonempty set of objects on which two operations are defined: addition, and multiplication by scalars. By ***addition*** we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in V an object $\mathbf{u}+\mathbf{v}$, called the ***sum*** of $\mathbf{u}$ and $\mathbf{v}$; by ***scalar multiplication*** we mean a rule for associating with each scalar k and each object $\mathbf{u}$ in V an object $k\mathbf{u}$, called the ***scalar multiple*** of $\mathbf{u}$ by k. If the following axioms are satisfied by all objects $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V and all scalars k and m, then we call V a ***vector space*** and we call the objects in V ***vectors***.

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u}+\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
4. There is an object $\mathbf{0}$ in V, called a ***zero vector*** for V, such that $\mathbf{0}+\mathbf{u}=\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}$ in V.
5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a ***negative*** of $\mathbf{u}$, such that $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$.
6. If k is any scalar and $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V.
7. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
8. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$
9. $k(m\mathbf{u})=(km)(\mathbf{u})$
10. $1\mathbf{u}=\mathbf{u}$

Vector space scalars can be real numbers or complex numbers. Vector spaces with real scalars are called ***real vector spaces*** and those with complex scalars are called ***complex vector spaces***. For now we will be concerned exclusively with real vector spaces. We will consider complex vector spaces later.

Observe that the definition of a vector space does not specify the nature of the vectors or the operations. Any kind of object can be a vector, and the operations of addition and scalar multiplication need not have any relationship to those on R^n. The only requirement is that the ten vector space axioms be satisfied. In the examples that follow we will use four basic steps to show that a set with two operations is a vector space.

To Show that a Set with Two Operations is a Vector Space

Step 1. Identify the set V of objects that will become vectors.

Step 2. Identify the addition and scalar multiplication operations on V.

Step 3. Verify Axioms 1 and 6; that is, adding two vectors in V produces a vector in V, and multiplying a vector in V by a scalar also produces a vector in V. Axiom 1 is called ***closure under addition***, and Axiom 6 is called ***closure under scalar multiplication***.

Step 4. Confirm that Axioms 2, 3, 4, 5, 7, 8, 9, and 10 hold.

Historical Note The notion of an "abstract vector space" evolved over many years and had many contributors. The idea crystallized with the work of the German mathematician H. G. Grassmann, who published a paper in 1862 in which he considered abstract systems of unspecified elements on which he defined formal operations of addition and scalar mul-

Example 1.- $S = \{\boxslash\}$ a set of one element

with the two operations:

i) $\boxslash + \boxslash = \boxslash$, $\quad c\,\boxslash = \boxslash$.

Show that this set with these two operations is a vector space.

Proof.- - We define zero vector $= \boxslash$

- clearly $\boxslash$ is also its own negative because

$\boxslash + \boxslash = \boxslash$ by definition.

- $(c+d)\boxslash = \boxslash$ by definition

and $c\,\boxslash = 0$, $d\,\boxslash = 0$ also by def.

- $1\,\boxslash = \boxslash$ by def.

All the other axioms are also satisfied.

Example 2.-

$F(-\infty,\infty) = \{\vec{f} = f(x) : f : \mathbb{R} \to \mathbb{R}\}$ With the usual operations

i) $(\vec{f}+\vec{g})(x) = f(x) + g(x)$.

ii) $(k\vec{f})(x) = kf(x)$.

Zero Vector: $\vec{O}(x) = 0$, for all x.

negative of f: $(-\vec{f})(x) = -f(x)$, for all x.

All axioms are satisfied using properties of real numbers.

Other possibilities of Similar Vector spaces are

$F[a,b] = \{\vec{f} = f : f : [a,b] \longrightarrow \mathbb{R}\}$ and $F(a,b) = \{\vec{f} = f : f : (a,b) \to \mathbb{R}\}$.

Example 3.-

$\mathbb{P}_n = \{$ set of all polinomials of degree $\leq n\}$.

$\mathbb{P}_n = \{\vec{p} = p(t) : p(t) = a_0 + a_1 t + \dots + a_n t^n, a_0, \dots, a_n, t \in \mathbb{R}\}$.

With the usual operations:

$(\vec{p}+\vec{q})(t) = p(t) + q(t)$

$(c\vec{p})(t) = c\,p(t)$.

Again, it can be easily proved that all axioms are satisfied from properties of real numbers.

Example 4.- Show that the set

$$S = \{ \vec{u} \in \mathbb{R}^2 : \vec{u} = (u_1, u_2), \ u_1, u_2 \in \mathbb{R} \}$$

with the operations

i) $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2)$.

ii) $k\vec{u} = (ku_1, 0)$

is not a vector space.

All axioms are satisfied except

$$1\vec{u} = 1(u_1, u_2) = (1u_1, 0) = (u_1, 0) \neq (u_1, u_2) = \vec{u}.$$

Show the rest.

Example 5 — INTERESTING VECTOR SPACE

$V = \mathbb{R}^+ = (0, +\infty)$

Start with these

Operations:
1) $u + v \stackrel{def}{=} uv$
2) $ku = u^k$

then $\boxed{1+1 = 1.1 = 1}$ and $\boxed{2(1) = 1^2 = 1}$

All axioms are verified!

i) $\vec{0}$ vector is 1 or $\vec{0} = 1$.

ii) for u, $-u = \frac{1}{u}$ (Notice that "0" is not included in V)

because $u + (-u) = u\frac{1}{u} = 1 = \vec{0}$.

iii) $k(u+v) = (uv)^k = u^k v^k = (ku)+(kv)$

iv) $u + (v+w) = (u+v)+w$

Ask students to prove this → Since $u + (v+w) = u(v+w) = u(vw) = (uv)w =$
$= (u+v)w = (u+v)+w$

Example 6

A Set of two elements

$S = \{ \text{moon}, \text{Sun} \}$ Cannot be a Vector space.

Make all possible addition and multiplication by Scalars definitions and Show that the 10 axioms are not all Satisfied in each case.

Example 7.-

The set $S = \{ \text{moon}, \text{Sun}, \text{earth} \}$ of three elements form a Vector space for appropriate definitions Of addition and Scalar multiplication operations.

Define appropriate operations and Show that S is a Vector space under these operations.

Example 8.-

$$V = \left\{ \begin{array}{l} \text{Set of matrices } 2\times 2 \text{ with usual operations of} \\ \text{matrix addition and scalar multiplication} \end{array} \right\}$$

$$= \left\{ \vec{u} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} : \quad u_{ij} \in \mathbb{R}, \quad i,j = 1,2 \right\}$$

Operations are closed. In fact,

i) Additions:

$$\vec{u} + \vec{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$ is in V

ii) Multiplication by Scalars

$$k\vec{u} = k \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$ is in V

Normally, the vector space V above is denoted by

$$V = M_{2\times 2}.$$

SUBSPACES.

We know that the set $S = \text{span}\{\vec{v}_1, \vec{v}_2\}$, for $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^3$ and $\vec{v}_1 \neq \vec{0}$ or $\vec{v}_2 \neq \vec{0}$, defines a line or a plane in $\mathbb{R}^3$.

It can be proved that S is a vector space (prove it!) and since $S \subset \mathbb{R}^3$, it is said that S is a subspace of $\mathbb{R}^3$ under the same operations defined in $\mathbb{R}^3$.

Definition

A subset W of a vector space V is called a subspace of V if W is itself a vector space under the addition and scalar multiplication defined on V.

All axioms of V are inherited for elements of W except Axioms 1, 4, 5, and 6. Why?

Theorem.– $W \subset V$ contains at least one vector.

W is a subspace of V if and only if

a) $\vec{u}, \vec{v} \in W \Rightarrow \vec{u} + \vec{v} \in W$. "closed under addition"

b) $k \in \mathbb{R}$, $\vec{u} \in W \Rightarrow k\vec{u} \in W$. "closed under scalar multiplication".

Proof.– $(\leftarrow)$

We only need to verify two axioms:

i) $\vec{0} \in W$;

ii) for every $\vec{u} \in W$, the negative $-\vec{u}$ is also in W.

Now, (i) $\vec{0} \in W$, since

$k\vec{u} \in W$ for any $k \in \mathbb{R}$. In particular,

for $k = 0$ $\quad k\vec{u} = 0\vec{u} = \vec{0}$ (proof in homework).

Also

If $\vec{u} \in W$, then $(-1)\vec{u} \in W$, but

$(-1)\vec{u} = -\vec{u}$ (proof in homework).

All the other axioms are also satisfied in W because they are satisfied in V for the same two operations.

$(\rightarrow)$ If W is a subspace then it already satisfies (a) and (b).

<u>Def.</u>— For any set $\{\vec{v}_1, .., \vec{v}_p\} \subset V$ (vector space) we call $\mathrm{Span}\{\vec{v}_1, .., \vec{v}_p\}$, the set of all vectors that can be written as linear combinations of $\vec{v}_1, .., \vec{v}_p$.

<u>Thm 1.</u>— $H = \mathrm{Span}\{\vec{v}_1, .., \vec{v}_p\}$ is a subspace of V.

<u>Proof.</u>—

For $\vec{u}, \vec{v} \in H$

$$\vec{u} = s_1\vec{v}_1 + \cdot\cdot + s_p\vec{v}_p, \qquad \vec{v} = t_1\vec{v}_1 + \cdot\cdot + t_p\vec{v}_p$$

using properties of $\mathbb{R}$ for addition and scalar multiplication.

then $\vec{u} + \vec{v} = (s_1 + t_1)\vec{v}_1 + \cdot\cdot + (s_p + t_p)\vec{v}_p \in H$. Closed under addition

Also $k\vec{u} = k(s_1\vec{v}_1 + \cdot\cdot + s_p\vec{v}_p) = (ks_1)\vec{v}_1 + \cdot\cdot + (ks_p)\vec{v}_p \in H$.

Closed under scalar multiplic.

Some exercises of Section 4.1

#6 $S = \{ \vec{p} : \vec{p}^{(t)} = p(t) = a + t^2;\ a \in \mathbb{R} \}$.

Not a Subspace, since

a) $(-1)\,\vec{p}(t) = (-1)\,p(t) = -a - t^2 \notin S$.

or

b) $\vec{0}(t) = 0(t) = 0 + 0t + 0t^2 \notin S$.

#8 $S = \{ \vec{p} \in \mathbb{P}_n : \vec{p}(0) = p(0) = 0 \} = \{ \vec{p} \in \mathbb{P}_n : p(t) = a_1 t + \cdots + a_n t^n,\ a_1, \ldots a_n \in \mathbb{R}. \}$

This a Subspace of $\mathbb{P}_n$

Since if $\vec{p}$ and $\vec{q} \in S \Rightarrow (\vec{p} + \vec{q})(0) = (p+q)(0) = p(0) + q(0) = 0 + 0 = 0$

Also $c\vec{p}(0) = c\,p(0) = c \cdot 0 = 0$

therefore, S is closed under addition and also under scalar multiplication. As a consequence, S is a subspace of $\mathbb{P}_n$.

#13) c) $\vec{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$

Is $\vec{w}$ in Span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

equivalent to say: are there c_1, c_2, c_3 such that

$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{w}$ or $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ -1 & 3 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$?

#20) b) Is $W = \{ \vec{f} \in C[a,b] : \vec{f}(a) = \vec{f}(b) \}$ a Subspace?

If $\vec{f}$ and $\vec{g}$ are in W, then $\vec{f}(a) = \vec{f}(b) \Rightarrow f(a) = f(b)$
$\vec{g}(a) = \vec{g}(b) \Rightarrow g(a) = g(b)$.

i) thus, $(\vec{f}+\vec{g})(a) = (f+g)(a) = f(a) + g(a) = f(b) + g(b)$
$= \vec{f}(b) + \vec{g}(b) = (\vec{f}+\vec{g})(b)$.

ii) Also $(c\vec{f})(a) = (cf)(a) = c\,f(a) = c\,f(b) = (cf)(b) = (c\vec{f})(b)$

Since W is clearly a nonempty subset of $C[a,b]$.
and Satisfies (i) and (ii) then W is a Subspace
of $C[a,b]$.

#16, 17 Find a set S of Vectors that Spans W or
Show why W is not a vector space.

16) $W = \left\{ \vec{v} \in \mathbb{R}^3 : \vec{v} = \begin{bmatrix} 1 \\ 3a-5b \\ 3b+2a \end{bmatrix} \right\}$

W is not a Subspace of $\mathbb{R}^3$. Many axioms fail.
For example,
If $\vec{v}, \vec{u} \in W$

$\Rightarrow \vec{v} + \vec{u} = \begin{bmatrix} 1+1 \\ *+* \\ *+* \end{bmatrix} = \begin{bmatrix} 2 \\ *+* \\ *+* \end{bmatrix} \notin W$.

#17 $W = \left\{ \vec{v} \in \mathbb{R}^3 : \vec{v} = \begin{bmatrix} 2a-b \\ 3b-c \\ 3c-a \\ 3b \end{bmatrix} a, b, c \text{ arbitrary real numbers} \right\}$

Notice that

$$\begin{bmatrix} 2a-b \\ 3b-c \\ 3c-a \\ 3b \end{bmatrix} = \begin{bmatrix} 2a \\ 0 \\ -a \\ 0 \end{bmatrix} + \begin{bmatrix} -b \\ 3b \\ 0 \\ 3b \end{bmatrix} + \begin{bmatrix} 0 \\ -c \\ 3 \\ 0 \end{bmatrix} =$$

$$= a \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 3 \\ 0 \\ 3 \end{bmatrix} + c \begin{bmatrix} 0 \\ -1 \\ 3 \\ 0 \end{bmatrix}$$

Thece $W = \text{Span}\left\{ \begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 3 \\ 0 \end{pmatrix} \right\}$

is a Subspace of $\mathbb{R}^4$!

Some Proofs in Section 4.1

Thm

Section 4.1

Exercises # 27, 28, 29

V is a vector space, $\vec{u}$ and $\vec{v}$ are in V and k is scalar, then

a) $0\vec{u} = 0$, b) $k\vec{0} = \vec{0}$, c) $(-1)\vec{u} = -\vec{u}$.

d) If $k\vec{u} = \vec{0}$, then $k = 0$ or $\vec{u} = \vec{0}$.

Proof - First, keep in mind that all axioms are satisfied

#28

b) $k\vec{0} = \vec{0}$

$$k\vec{0} \overset{④}{=} k(\vec{0}+\vec{0}) \overset{⑦}{=} k\vec{0} + k\vec{0} \qquad (*)$$

Now, $-k\vec{0}$ exists (axiom 5). Adding it to both sides of equation (*).

$$\underbrace{k\vec{0} + (-k\vec{0})}_{Ⓐ} = \underbrace{(k\vec{0} + k\vec{0}) + (-k\vec{0})}_{Ⓑ}$$

Now, $Ⓐ = k\vec{0} + (-k\vec{0}) \overset{⑤}{=} \vec{0}$

and $Ⓑ \overset{③}{=} k\vec{0} + (k\vec{0} + (-k\vec{0})) \overset{⑤}{=} k\vec{0} + \vec{0} \overset{④}{=} k\vec{0}$.

Therefore, $Ⓐ = Ⓑ \Leftrightarrow \boxed{k\vec{0} = \vec{0}}$ ✓

thm

Section 4.1

#27 a) $0\vec{u} = \vec{0}$

Proof.

$$0\vec{u} = (0+0)\vec{u} \overset{⑧}{=} 0\vec{u} + 0\vec{u}$$

Adding $-0\vec{u}$ ⑤ to both sides

$$\underbrace{0\vec{u} + (-0\vec{u})}_{A} = \underbrace{(0\vec{u} + 0\vec{u}) + (-0\vec{u})}_{B}$$

$$\begin{aligned}
A &\overset{⑤}{=} \vec{0} = 0\vec{u} + (0\vec{u} + (-0\vec{u})) \quad ③ \text{ (from } B)\\
\vec{0} &= 0\vec{u} + \vec{0} \quad ⑤\\
\vec{0} &= 0\vec{u} \quad ④
\end{aligned}$$

Section 4.1

#29 c) $(-1)\vec{u} = -\vec{u}$

Proof.

$$\vec{u} + (-1)\vec{u} \overset{⑩}{=} 1\vec{u} + (-1)\vec{u} \underset{⑧}{=} (1+(-1))\vec{u} \underset{\text{property of real #s}}{=} 0\vec{u} \underset{\text{ⓐ thm}}{=} \vec{0}$$

then

$$\vec{u} + (-1)\vec{u} = \vec{0}$$

adding $-\vec{u}$ in both sides

$$\begin{aligned}
-\vec{u} + (\vec{u} + (-1)\vec{u}) &= -\vec{u} + \vec{0} \\
\overset{③}{\parallel} \qquad & \quad \overset{④}{\parallel} \\
(-\vec{u} + \vec{u}) + (-1)\vec{u} &= -\vec{u} \\
\overset{⑤}{\parallel} \qquad & \\
\vec{0} + (-1)\vec{u} &= -\vec{u} \quad \cdots \to \\
\overset{④}{\parallel} \qquad & \\
(-1)\vec{u} &= -\vec{u} \ ✓
\end{aligned}$$

(Good to emphasize logic)

d) If $k\vec{u}=\vec{0}$, then $k=0$ or $\vec{u}=\vec{0}$.

Section 4.1 #30

Proof.-

Assume $k\vec{u}=\vec{0}$, (1)

then there are two possibilities for k:

a) $k=0$ and the theorem is proved.

or

b) $k\neq 0$, then $\frac{1}{k}$ is a scalar too

and multiplying both sides of (1) by it

$$\underbrace{\frac{1}{k}(k\vec{u})}_{A}=\underbrace{\frac{1}{k}\vec{0}}_{B}$$

Now, $A \overset{9}{=} \left(\frac{1}{k}k\right)\vec{u} = 1\vec{u} \overset{10}{=} \vec{u}$

and $B = \frac{1}{k}\vec{0} \overset{(b)}{=} \vec{0}$

Therefore, $A=B \Leftrightarrow \boxed{\vec{u}=\vec{0}.}$

Uniqness Theorem.

Thm.- Given a vector $\vec{u}$ in a vector space V, the negative vector $-\vec{u}$ is unique.

Section 4.1

#26

Proof.- If there were another vector $\vec{v}$ in V such that

$$\vec{u}+\vec{v}=\vec{0}, \qquad (1)$$

then adding $-\vec{u}$ to both sides of (1)

$$\underbrace{-\vec{u}+(\vec{u}+\vec{v})}_{\text{(A)}} = \underbrace{-\vec{u}+\vec{0}}_{\text{(B)}}$$

Now,

$$\text{(A)} = -\vec{u}+(\vec{u}+\vec{v}) \overset{(3)}{=} (-\vec{u}+\vec{u})+\vec{v} \overset{(5)}{=} \vec{0}+\vec{v} \overset{(4)}{=} \vec{v}$$

and

$$\text{(B)} = -\vec{u}+\vec{0} \overset{(4)}{=} -\vec{u}$$

Therefore $\text{(A)}=\text{(B)} \Rightarrow \boxed{\vec{v}=-\vec{u}}$

And as a consequence $-\vec{u}$ is unique.